J Math Chem (2015) 53:880–892
DOI 10.1007/s10910-014-0460-8

ORIGINAL PAPER

Determination of multiple roots of nonlinear equations and applications

José L. Hueso · Eulalia Martínez · Carles Teruel

Received: 15 July 2014 / Accepted: 15 December 2014 / Published online: 24 December 2014

Abstract In this work we focus on the problem of approximating multiple roots of nonlinear equations. Multiple roots appear in some applications such as the compression of band-limited signals and the multipactor effect in electronic devices. We present a new family of iterative methods for multiple roots whose multiplicity is known. The methods are optimal in Kung–Traub's sense (Kung and Traub in J Assoc Comput Mach 21:643–651, [1]), because only three functional values per iteration are computed. By adding just one more function evaluation we make this family derivative free while preserving the convergence order. To check the theoretical results, we codify the new algorithms and apply them to different numerical examples.

Keywords Iterative methods · Nonlinear equations · Multiple roots · Convergence order · Efficiency

1 Introduction

Solving nonlinear equations is an important problem in applied mathematics and engineering. In this work, we apply iterative methods for finding a zero of a continuously

This research was supported by Ministerio de Ciencia y Tecnología MTM2011-28636-C02-02 and by Vicerrectorado de Investigación, Universitat Politècnica de València PAID-SP-2012-0474.

J. L. Hueso (✉) · C. Teruel
Instituto Universitario de Matemática Multidisciplinar, Universitat Politècnica de València, Valencia, Spain
e-mail: jlhueso@mat.upv.es

E. Martínez
Instituto Universitario de Matemática Pura y Aplicada,
Universitat Politècnica de València, Valencia, Spain

differentiable function $f : \mathbb{R} \longrightarrow \mathbb{R}$ that is, a solution α of the nonlinear equation $f(x) = 0$.

In the case of simple roots, many robust and efficient methods have been proposed with high convergence order (see [2–4]).

Here we focus in the case of a root α of multiplicity $m > 1$, namely, $f(\alpha) = 0$, $f^{(k)}(\alpha) = 0$ for $k = 1, \dots, m-1$ and $f^{(m)}(\alpha) \neq 0$. It is well known that the convergence order of iterative methods decreases in the presence of a multiple root. In this sense, modifications in the iterative function can improve the behavior of the method. Newton's method recovers the second order convergence for multiple roots, [5], with the modification given by

$$x_{k+1} = x_k - m\frac{f(x_k)}{f'(x_k)}.$$

Recently, some authors [6–8] have obtained higher order iterative methods for multiple roots when the multiplicity is known in advance.

Our aim in this study is, first of all, to present some new efficient iterative methods for approximating multiple roots. For this purpose we introduce some parameters in optimal iterative methods for simple roots and use the relations deduced from the fact of being a multiple root for getting the value of the parameters that re-establishes the convergence order.

Secondly, we are interested in obtaining derivative free iterative methods for multiple roots. In the literature, some high order methods avoiding the use of derivatives have been presented for the case of simple roots, [9–13]. To our knowledge, the use of divided differences for approximating the derivative in the case of multiple roots has not been published yet. We use suitable divided differences, showing that they preserve the convergence order. To check the theoretical results, we codify the new algorithms and apply them to different numerical tests.

The determination of multiple roots is of interest in some branches of applied sciences. For example, in [14] the authors show that the zeros of the derivative of a band-limited signal that goes through an unknown singularity can be useful in identifying and correcting it. In particular, they prove that the the bandwidth of a signal can be compressed by a ratio of $1/n$ if and only if the signal has roots of multiplicity n.

Multipactor is an undesirable RF breakdown that may occur in the high power microwave devices working under the vacuum condition [15]. A particular scenario where the multipactor appears is inside a parallel plate waveguide. Between these two plates, there exists an electric field with an electric potential difference which produces the electron movement. In the study of the electron trajectories, a case of interest is when the electron reaches a plate with zero speed. In this case, the function distance from the electron to the plate presents a zero of multiplicity 2.

Nonlinear systems also appear in Chemistry, for example in the study of stability of chemical reactions. Here we deal with the solution of Van der Waals' equation for the determination of the volume of a gas in the particular case when the solutions are multiple.

The rest of the paper is organized as follows. In Sect. 2 the construction of the family is explained and the convergence result is proved. Derivative free iterative methods are stated in Sect. 3. In Sect. 4 we compare the new methods with an existing fourth order method by using example equations with multiple roots. Section 5 is devoted to the conclusions.

2 Developing new iterative methods

In [16], we have presented new families of iterative methods for nonlinear systems. Here we use the same scheme, but now the determination of parameters will be done assuming that the root is of multiplicity m. With the notation for the unidimensional case, the proposed family can be written as:

$$\begin{aligned} y_n &= x_n - b\frac{f(x_n)}{f'(x_n)}, \\ x_{n+1} &= x_n - \left(s_1 + s_2 h(y_n, x_n) + s_3 h(x_n, y_n) + s_4 h(y_n, x_n)^2\right)\frac{f(x_n)}{f'(x_n)}, \end{aligned} \tag{1}$$

where $h(x_n, y_n) = \frac{f'(y_n)}{f'(x_n)}$, and $b, s_1, s_2, s_3, s_4 \in \mathbb{R}$.

Now we determine the value of these real parameters in order to obtain the maximum efficiency. According to Kung–Traub's conjecture, [1], as only three functional evaluations per iteration are used, the optimal order will be 4. Consider the Taylor's expansion of $f(x_n)$ around the solution α

$$f(x_n) = \frac{f^{(m)}(\alpha)}{m!} e_n^m \left(\sum_{i=0}^{4} c_i e_n^i + O\left(e_n^5\right)\right), \tag{2}$$

where $e_n = x_n - \alpha$ and $c_i = \frac{m!}{(m+i)!}\frac{f^{(m+i)}(\alpha)}{f^{(m)}(\alpha)}$, $i \geq 0$. Then, the derivative is

$$f'(x_n) = \frac{f^{(m)}(\alpha)}{m!} e_n^{m-1} \left(\sum_{i=0}^{4} (m+i) c_i e_n^i + O\left(e_n^5\right)\right). \tag{3}$$

So, the error equation for a Newton's step can be written as

$$\begin{aligned} \hat{e}_n = y_n - \alpha = e_n - b\frac{f(x_n)}{f'(x_n)} &= \left(1 - \frac{b}{m}\right) e_n + \frac{bc_1}{m^2} e_n^2 - \frac{b\left((1+m)c_1^2 - 2mc_2\right)}{m^3} e_n^3 \\ &+ \frac{b\left((1+m)^2 c_1^3 - m(4+3m)c_1 c_2 + 3m^2 c_3\right)}{m^4} e_n^4 + O\left(e_n^5\right), \end{aligned}$$

and then, $f'(y_n)$ takes the form

$$f'(y_n) = \frac{f^{(m)}(\alpha)}{m!} \hat{e}_n^{m-1} \left(\sum_{i=0}^{4} (m+i) c_i \hat{e}_n^i + O\left(\hat{e}_n^5\right)\right).$$

By using Mathematica, we compute $h(x_n, y_n)$ and $h(y_n, x_n)$ in terms of e_n obtaining the following error equation for (1):

$$e_{n+1} = A_1 e_n + A_2 e_n^2 + A_3 e_n^3 + A_4 e_n^4 + O(e_n)^5, \tag{4}$$

where

$$A_1 = 1 - \frac{s_1 + s_4 \mu^{2-2m} + s_2 \mu^{1-m} + s_3 \mu^{-1+m}}{m},$$
$$A_2 = \Big(m^2 s_1 + s_4 \mu^{2-2m} + s_2 \mu^{1-m} + s_3 \mu^{-1+m} - b(b-2m+bm)\mu^{-2m}\left(\left(-2+\frac{2b}{m}\right)s_4 - s_2\mu^m + \frac{m^2 s_3 \mu^{3m}}{(b-m)^2}\right)\Big)\frac{c_1}{m^4}$$

and $\mu = 1 - \frac{b}{m}$.

Now, in order to annihilate the terms of first and second order in (4), we solve the system $A_1 = 0$, $A_2 = 0$ and find the parameters s_1 and s_2:

$$s_1 = -\frac{\mu^{-1-2m}}{b(b-2m+bm)}\Big(\mu^{2+2m}\Big(-m^2 s_4 \mu^{2-2m} - m^2 s_3 \mu^{-1+m} + \frac{2b(b-m)(b-2m+bm)s_4\mu^{-2m}}{m} + \frac{bm^2(b-2m+bm)s_3\mu^m}{(b-m)^2}\Big) + \left(-2bm + b^2(1+m) + m^2\mu\right)\left(s_4\mu^3 + \mu^{2m}\left(-m\mu + s_3\mu^m\right)\right)\Big), \tag{5}$$

$$s_2 = \frac{\mu^{-m}}{bm(-b+m)^2(b-2m+bm)}\Big(2b^5(1+m)s_4 + 6b^3m^2(3+m)s_4 - 2b^4m(5+3m)s_4 - m^6\mu^{2m} + 2bm^4\left(2s_4 + \mu^{2m}\left(m - s_3\mu^m\right)\right) + b^2m^3\left(-2(7+m)s_4 + \mu^{2m}\left(-m + s_3\mu^m + ms_3\mu^m\right)\right)\Big). \tag{6}$$

By substituting these values in (4), we have

$$e_{n+1} = (k(b-m)^4 m^5(-2m + b(2+m))\mu^{2m} c_2 + B)e_n^3 + O\left(e_n^4\right),$$

with $k, B \in \mathbb{R}$. So, in order to eliminate the coefficient of c_2 in the term of third order, we impose the condition $b = \frac{2m}{2+m}$, and then one has $\mu = \frac{m}{2+m}$, so that the coefficient of e_n^3 becomes

$$A_3 = -\frac{2\mu^{-2m}}{m^6(2+m)^4}\Big(-8m^4\mu^{2m} - 12m^5\mu^{2m} - 6m^6\mu^{2m} - m^7\mu^{2m} + 64s_3\mu^{3m} + 96ms_3\mu^{3m} + 48m^2s_3\mu^{3m} + 8m^3\left(s_4 + s_3\mu^{3m}\right)\Big)c_1^2.$$

We obtain s_3 in order to annihilate this term, obtaining

$$s_3 = \frac{m^3\mu^{-3m}\left(-8s_4 + 8m\mu^{2m} + 12m^2\mu^{2m} + 6m^3\mu^{2m} + m^4\mu^{2m}\right)}{8(2+m)^3}. \tag{7}$$

Finally, substituting (5–7) into (4), the error equation of the family of iterative methods defined by (1) is

$$e_{n+1} = \frac{1}{3m^5(2+m)^5}\left(\left(128m + 288m^2 + 352m^3 + 368m^4 + 312m^5 + 178m^6 + 62m^7 + 12m^8 + m^9 - 192s_4\right)c_1^3 - 3m^4(2+m)^5c_1c_2 + 3m^6(2+m)^3c_3\right)e_n^4 + O\left(e_n^5\right), \tag{8}$$

and so, it has fourth convergence order. Then, we have proved the following result:

Theorem 1 Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be a sufficiently differentiable function in an open interval I containing α, that is a root of multiplicity m of the equation $f(x) = 0$. Then, for an initial approximation sufficiently close to α, the family of methods defined by (1) where $b = \frac{2m}{2+m}$ and s_1, s_2 and s_3 take the values given by (5), (6) and (7), respectively, has fourth order of convergence for any $s_4 \in \mathbb{R}$. The error equation of the family is given by (8).

3 Derivative free methods for multiple roots

In order to avoid the use of the use of derivatives we propose the approximation by divided differences given by:

$$f'(x_n) \approx f[z_n, x_n] = \frac{f(z_n) - f(x_n)}{z_n - x_n}, \tag{9}$$

where $z_n = x_n + f(x_n)^q$ with $q \geq 1$. In [17], the authors prove that approximating the derivative by (9) the convergence order of many iterative methods for simple roots remains invariant.

Our aim here is to extend this technique to the case of multiple roots. Then, replacing in the family (1) the derivatives with divided differences, one obtains the derivative free family given by:

$$\begin{aligned} \tilde{y}_n &= x_n - b\frac{f(x_n)}{f[x_n + f(x_n)^q, x_n]} \\ \tilde{x}_{n+1} &= x_n - \left(s_1 + s_2\tilde{h}(\tilde{y}_n, x_n) + s_3\tilde{h}(x_n, \tilde{y}_n) + s_4\tilde{h}(\tilde{y}_n, x_n)^2\right)\frac{f(x_n)}{f[x_n + f(x_n)^q, x_n]} \end{aligned} \tag{10}$$

where $\tilde{h}(x_n, \tilde{y}_n) = \frac{f[\tilde{y}_n + f(\tilde{y}_n)^q, \tilde{y}_n]}{f[x_n + f(x_n)^q, x_n]}$. We can prove the following result:

Theorem 2 Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be a sufficiently differentiable function in an open interval I containing α, that is a root of multiplicity m of the equation $f(x) = 0$. Then, for an initial approximation sufficiently close to α, the family of methods defined by (10) with $s_4 \in \mathbb{R}$, and the same values of the parameters b, s_1, s_2 and s_3 that have been defined in (1) has the following error equation:

$$\tilde{e}_{n+1} = e_{n+1} + O\left(e_n^{qm}\right) \tag{11}$$

where e_{n+1} is the error equation obtained in (8) for the method that uses derivatives given by (1).

Proof It is well known that the forward approximation for the derivative is of order h, that is:

$$f[x, x+h] = \frac{f(x+h) - f(x)}{h} = f'(x) + O(h),$$

and consequently for the inverse operator we have:

$$\frac{1}{f[x, x+h]} = \frac{1}{f'(x)} + O(h). \tag{12}$$

But, specifically by the corresponding Taylor's development we can express:

$$f[x, x+h] = \frac{f(x+h) - f(x)}{h} = f'(x) + \frac{f''(x)}{2}h + O(h^2),$$

then, for multiple roots, setting $x = x_n$ and $h = f(x_n)^q$ by taking into account (2), we have $h = O\left(e_n^{qm}\right)$ and $f''(x_n) = O\left(e_n^{m-2}\right)$ so, we get an approximation for $f'(x_n)$ verifying:

$$f[x_n + f(x_n)^q, x_n] = f'(x_n) + O\left(e_n^{(q+1)m-2}\right) \tag{13}$$

and by (12) we obtain:

$$\frac{1}{f[x_n + f(x_n)^q, x_n]} = \frac{1}{f'(x_n)} + O\left(e_n^{qm}\right), \tag{14}$$

then, by using (2) we have:

$$\frac{f(x_n)}{f[x_n + f(x_n)^q, x_n]} = \frac{f(x_n)}{f'(x_n)} + O\left(e_n^{(q+1)m}\right). \tag{15}$$

This fact allows us to establish the relation between the first step in (1) and (10) by using (12) and (13) one has:

$$\begin{aligned}\tilde{y}_n &= x_n - b\frac{f(x_n)}{f[x_n + f(x_n)^q, x_n]} = x_n - b\left(\frac{f(x_n)}{f'(x_n)} + O\left(e_n^{(q+1)m}\right)\right)\\ &= y_n + O\left(e_n^{(q+1)m}\right),\end{aligned}$$

and consequently $f(\tilde{y}_n) = f(y_n) + O\left(e_n^{(q+1)m}\right)$ and $f'(\tilde{y}_n) = f'(y_n) + O\left(e_n^{(q+1)m}\right)$.

Now, we analyze the function $\tilde{h}(x_n, \tilde{y}_n)$ appearing in the second step of (10), obtaining that:

$$
\begin{aligned}
\tilde{h}(x_n, \tilde{y}_n) &= \frac{f[\tilde{y}_n + f(\tilde{y}_n)^2, \tilde{y}_n]}{f[x_n + f(x_n)^2, x_n]} = \left(f'(\tilde{y}_n) + O\left(e_n^{(q+1)m-2}\right)\right)\left(\frac{1}{f'(x_n)} + O\left(e_n^{qm}\right)\right) \\
&= \left(f'(y_n) + O\left(e_n^{(q+1)m-2}\right)\right)\left(\frac{1}{f'(x_n)} + O\left(e_n^{qm}\right)\right) = \frac{f'(y_n)}{f'(x_n)} + O\left(e_n^{qm-1}\right) \\
&= h(x_n, y_n) + O\left(e_n^{qm-1}\right),
\end{aligned}
$$

Analogously, we have the following:

$$
\tilde{h}(\tilde{y}_n, x_n) = h(y_n, x_n) + O\left(e_n^{qm-1}\right). \tag{16}
$$

Now, analyzing the relation between the second step in (1) and in (10), by using (15) and (16) we get:

$$
\begin{aligned}
\tilde{e}_{n+1} &= \tilde{x}_{n+1} - \alpha = x_n - \alpha \\
&\quad - \left(s_1 + s_2\tilde{h}(\tilde{y}_n, x_n) + s_3\tilde{h}(x_n, \tilde{y}_n) + s_4\tilde{h}(\tilde{y}_n, x_n)^2\right)\left(\frac{f(x_n)}{f[x_n + f(x_n)^q, x_n]}\right) \\
&= x_n - \alpha - \left(s_1 + s_2 h(y_n, x_n) + s_3 h(x_n, y_n) + s_4 h(y_n, x_n)^2 + O\left(e_n^{qm-1}\right)\right) \\
&\quad \times \left(\frac{f(x_n)}{f'(x_n)} + O\left(e_n^{(q+1)m}\right)\right) = x_{n+1} - \alpha + O\left(e_n^{qm}\right) = e_{n+1} + O\left(e_n^{qm}\right) \\
&= O\left(e_n^4\right) + O\left(e_n^{qm}\right)
\end{aligned} \tag{17}
$$

where we have applied that e_{n+1} is of order 4, with the error equation obtained in (8). □

Note: We conclude that if $q = 1$ the derivative free iterative methods, (10), preserve the convergence order if $m \geq 4$ however for $q \geq 2$ the convergence order is maintained for all $m \geq 2$.

That is for for problems with multiple roots with multiplicity $m \geq 4$ approximating the derivative by

$$
f[x, x + f(x)] = \frac{f(x + f(x)) - f(x)}{f(x)} \tag{18}
$$

the iterative methods presented reach convergence order four but, if the multiplicity is $m < 4$ for preserving the convergence order we use:

$$
f[x, x + f(x)^2] = \frac{f(x + f(x)^2) - f(x)}{f(x)^2} \tag{19}
$$

4 Numerical tests

In this section, we choose specific values of the parameter s_4 in order to apply the methods to some examples. Remember the general expression of the family of iterative methods:

$$
\begin{aligned}
y_n &= x_n - b\frac{f(x_n)}{f'(x_n)},\\
x_{n+1} &= x_n - \left(s_1 + s_2 h(y_n, x_n) + s_3 h(x_n, y_n) + s_4 h(y_n, x_n)^2\right)\frac{f(x_n)}{f'(x_n)},
\end{aligned} \tag{20}
$$

where $h(x_n, y_n) = \frac{f'(y_n)}{f'(x_n)}$, $b = \frac{2m}{2+m} = 2\mu$, and s_1, s_2 and s_3 are given by (5), (6) and (7), respectively. The simplest element of the family is obtained for $s_4 = 0$. The values of the parameters for this method, that will be called MR_0 are given by:

$$
\begin{aligned}
s_1 &= -\frac{1}{4}m(-4 + 2m + 3m^2 + m^3),\\
s_2 &= \frac{1}{8}m\mu^m(2+m)^3,\\
s_3 &= \frac{1}{8}m^4\mu^{-m}.
\end{aligned}
$$

For $s_4 = 1$ we obtain the method called MR_1 whose values for the parameters are:

$$
\begin{aligned}
s_1 &= \frac{m\left(16 - 16m^2 - 18m^3 - 7m^4 - m^5 + m\left(8 + 12\mu^{-2m}\right)\right)}{4(2+m)^2},\\
s_2 &= \frac{1}{8}\mu^{1-m}\left(-24 + (2+m)^4\mu^{2m}\right),\\
s_3 &= \frac{m^3\mu^{-3m}\left(-8 + m(2+m)^3\mu^{2m}\right)}{8(2+m)^3}.
\end{aligned}
$$

The corresponding derivative free iterative methods obtained from (10) for these values of the parameters are denoted by DF_0^1 and DF_1^1 respectively when we use the divided difference given by (18) and DF_0^2 and DF_1^2 when the derivative is approximated by (19).

These methods will be compared with a fourth order method introduced in [18] that we denote by MR_{Sh}, and its derivative free versions by DF_{Sh}^1 and DF_{Sh}^2. The expression of the method MR_{Sh} is

$$
x_{n+1} = x_n - a_1 w_1(x_n) - a_2 w_2(x_n) - a_3\frac{w_2(x_n)^2}{w_1(x_n)},
$$

where $w_1(x_n) = \frac{f(x_n)}{f'(x_n)}$, $w_2(x_n) = \frac{f(x_n)}{f'(y_n)}$, $y_n = x_n - \beta w_1(x_n)$ and

$$
\begin{aligned}
\beta &= \frac{2m}{2+m},\\
a_1 &= \frac{1}{8}(m^3 - 4m + 8),\\
a_2 &= -\frac{1}{4}m(m-1)(m+2)^2\left(\frac{m}{2+m}\right)^m,
\end{aligned}
$$

$$a_3 = \frac{1}{8}m(m+2)^3\left(\frac{m}{2+m}\right)^{2m}.$$

We perform the calculations in Matlab (2011b) using variable precision arithmetic with 2,000 digits of mantissa. The stopping criterion is that the distance between consecutive iterates, $|x_{n+1} - x_n|$, is less than the tolerance 10^{-50} or that the function value is zero for the working precision. To measure the speed of convergence we use the approximated computational convergence order (see [13]),

$$ACOC = \frac{\log(|x_{k+1} - x_k|/|x_k - x_{k-1}|)}{\log(|x_k - x_{k-1}|/|x_{k-1} - x_{k-2}|)}.$$

In the study of the multipactor effect, the trajectory of an electron in the air gap between two parallel plates is given by

$$\begin{aligned} x(t) = x_0 + \left(v_0 + e\frac{E_0}{m\omega}\sin(\omega t_0 + \alpha)\right)(t - t_0) \\ + e\frac{E_0}{m\omega^2}\left(\cos(\omega t + \alpha) - \cos(\omega t_0 + \alpha)\right) \end{aligned} \tag{21}$$

where e and m are the charge and the mass of the electron at rest, x_0 and v_0 are the position and velocity of the electron at time t_0 and $E_0\sin(\omega t + \alpha)$ is the RF electric field between the plates.

Van der Waals' equation

$$\left(P + \frac{an^2}{V^2}\right)(V - nb) = nRT$$

explains the behavior of a real gas by introducing in the ideal gas equations two parameters, a and b, specific for each gas. The determination of the volume V of the gas in terms of the remaining parameters requires the solution of a nonlinear equation in V.

$$PV^3 - (nbP + nRT)V^2 + an^2V - an^2b = 0. \tag{22}$$

Given the constants a and b of a particular gas, one can find values for n, P and T, such that this equation has a triple root.

The first equation we consider is a particular case of (21) where the parameters have been normalized in order to deal with a simpler expression. The second one is an example of (22) where the root is triple. The next three equations appear in [18]. Equation f_4 is the particular case of $(x-1)^n(x-2)(x-3)$ for $n = 3$. For bigger values of n, the methods reach the desired convergence order, but their behaviors are very similar. The last equation contains a parameter that allows to obtain roots of arbitrary multiplicity. Two starting points are considered for each equation.

$f_1(x) = x + \cos(x) - \pi/2,$ $\alpha = \pi/2$, multiplicity: 3

Table 1 Convergence to the root $\alpha = \pi/2$ of multiplicity 3 of equation f_1

Method	x_0	n	$\|x_n - x_{n-1}\|$	*ACOC*	x_0	n	$\|x_n - x_{n-1}\|$	*ACOC*
MR_{Sh}	1	4	4.444e−121	5.0000	2	4	8.7412e−137	5.0000
MR_0		4	4.5571e−121	5.0000		4	8.8695e−137	5.0000
MR_1		4	4.5051e−121	5.0000		4	8.8106e−137	5.0000
DF^2_{Sh}	1	4	6.0505e−084	4.9951	2	4	6.5556e−103	4.9994
DF^2_0		4	6.0526e−084	4.9951		4	6.55e−103	4.9994
DF^2_1		4	6.0516e−084	4.9951		4	6.5525e−103	4.9994
DF^1_{Sh}	1	6	1.6353e−092	3.0000	2	6	3.753e−120	3.0000
DF^1_0		6	1.4209e−092	3.0000		6	3.5811e−120	3.0000
DF^1_1		6	1.5152e−092	3.0000		6	3.6587e−120	3.0000

$$f_2(V) = 0.98692V^3 - 5.18133V^2 + 9.06733V - 5.28927, \qquad \alpha = 1.75,\ \text{multiplicity: } 3$$

$$f_3(x) = x^2\exp(x) - \sin(x) + x, \qquad \alpha = 0,\ \text{multiplicity: } 2$$

$$f_4(x) = x^5 - 8x^4 + 24x^3 - 34x^2 + 23x - 6, \qquad \alpha = 1,\ \text{multiplicity: } 3$$

$$f_5(x) = (x^2 - e^x - 3x + 2)^5, \qquad \alpha \simeq 0.2575...,\ \text{multiplicity: } 5$$

$$f_6^n(x) = e^x - \sum_{k=0}^{n-1}\frac{x^k}{k!}, \qquad \alpha = 0,\ \text{multiplicity: } n.$$

The following tables show the starting points, the number of iterations, the latest increment and the ACOC for each test example for the considered methods. We denote by n.c. the cases where the method does not converge in 50 iterations.

The results for equation f_1 are shown in Table 1. The methods using derivatives have convergence order 5, instead of 4, because the coefficients $c_i = \frac{m!}{(m+i)!}\frac{f^{(m+i)}(\alpha)}{f^{(m)}(\alpha)}$ are 0 for $i = 1$ and $i = 3$, which annihilates the fourth order term in the error equation (8). The derivative free methods DF^2_{Sh}, DF^2_0 and DF^2_1 have the same convergence order and methods DF^1_{Sh}, DF^1_0 and DF^1_1 have convergence order 3, according to (17).

The three methods using derivatives reach the exact solution of the cubic equation f_2 in 2 steps, as shown in Table 2. The methods based on the divided differences (19) have a convergence order higher than expected because of the annihilation of some terms in the error expression (8). The derivative free methods introduced in this paper converge in less iterations than the corresponding Sharma type methods in this example.

Tables 3 and 4 show that the derivative free methods using the divided differences (18) have only convergence order 2 or 3 for roots of the same multiplicity, whereas the other methods, when converge, reach the fourth order convergence. The method MR_{Sh} behaves exactly as the method MR_1 for equation f_3, as it can be observed in Table 3, and so do their derivative free versions, DF^1_{Sh} and DF^2_{Sh} with respect to DF^1_1 and DF^2_1. This coincidence is not observed in the other examples.

For equation f_5, the last three methods in Table 5 do not converge starting from the considered points. Nevertheless, the first six methods show convergence of order 4.

Table 2 Convergence to the root $\alpha = 1.75$ of multiplicity 3 of equation f_2

Method	x_0	n	$\|x_n - x_{n-1}\|$	*ACOC*	x_0	n	$\|x_n - x_{n-1}\|$	*ACOC*
MR_{Sh}	0.51	2	0	–	3.1	2	0	–
MR_0		2	0	–	3.1	2	0	–
MR_1		2	0	–	3.1	2	0	–
DF^2_{Sh}	0.51	22	1.6369e−102	6.0000	3.1	11	0	6.0000
DF^2_0		11	8.2874e−307	6.0000	3.1	8	4.533e−163	6.0000
DF^2_1		13	2.1711e−274	6.0000	3.1	9	2.0209e−295	6.0000
DF^1_{Sh}	0.51	12	3.0051e−167	3.0000	3.1	13	3.7545e−230	3.0000
DF^1_0		10	2.1863e−173	3.0000	3.1	10	4.1962e−142	3.0000
DF^1_1		11	5.3937e−247	3.0000	3.1	11	1.4334e−177	3.0000

Table 3 Convergence to the root $\alpha = 0$ of multiplicity 2 of equation f_3

Method	x_0	n	$\|x_n - x_{n-1}\|$	*ACOC*	x_0	n	$\|x_n - x_{n-1}\|$	*ACOC*
MR_{Sh}	−0.5	5	5.7886e−056	3.9999	1	5	5.6183e−089	4.0000
MR_0		5	7.6979e−055	3.9999		5	2.5526e−085	4.0000
MR_1		5	5.7886e−056	3.9999		5	5.6183e−089	4.0000
DF^2_{Sh}	−0.5	6	1.1639e−175	4.0000	1	50	2.7752e−005	n.c.
DF^2_0		6	2.1411e−174	4.0000		50	5.9491e−005	n.c.
DF^2_1		6	1.1639e−175	4.0000		50	2.7752e−005	n.c.
DF^1_{Sh}	−0.5	9	1.0866e−096	2.0000	1	23	7.5738e−090	2.0000
DF^1_0		9	1.7357e−096	2.0000		20	1.4299e−069	2.0000
DF^1_1		9	1.0866e−096	2.0000		23	7.5738e−090	2.0000

Table 4 Convergence to the root $\alpha = 1$ of multiplicity 3 of equation f_4

Method	x_0	n	$\|x_n - x_{n-1}\|$	*ACOC*	x_0	n	$\|x_n - x_{n-1}\|$	*ACOC*
MR_{Sh}	0	5	6.2209e−101	4.0000	1.4	5	3.1888e−069	4.0000
MR_0		5	4.1156e−100	4.0000		5	6.006e−069	4.0000
MR_1		5	1.7444e−100	4.0000		5	4.5062e−069	4.0000
DF^2_{Sh}	0	50	1.2385e−005	n.c.	1.4	5	3.3419e−079	4.0000
DF^2_0		50	7.4313e−005	n.c.		5	1.8929e−078	4.0000
DF^2_1		50	4.4737e−005	n.c.		5	8.7317e−079	4.0000
DF^1_{Sh}	0	47	3.9625e−052	3.0000	1.4	6	2.4365e−094	3.0000
DF^1_0		18	2.7733e−144	3.0000		6	2.0752e−092	3.0000
DF^1_1		22	1.5767e−068	3.0000		6	2.8003e−093	3.0000

In the last example, setting the multiplicity $n = 6$, all the iterations converge with the desired order, as shown in Table 6.

The methods introduced in this paper present a similar performance as Sharma's method, or even better for the roots of higher multiplicity. In general the derivative free

Table 5 Convergence to the root $\alpha \simeq 0.2575$ of multiplicity 5 of equation f_5

Method	x_0	n	$\|x_n - x_{n-1}\|$	*ACOC*	x_0	n	$\|x_n - x_{n-1}\|$	*ACOC*
MR_{Sh}	0.15	4	8.1384e−099	4.0000	0.5	4	2.465e−075	4.0000
MR_0		4	7.8378e−099	4.0000		4	2.4315e−075	4.0000
MR_1		4	7.8777e−099	4.0000		4	2.436e−075	4.0000
DF^2_{Sh}	0.15	4	6.7771e−053	4.0001	0.5	7	1.0756e−193	4.0000
DF^2_0		4	6.7297e−053	4.0001		5	6.8349e−162	4.0000
DF^2_1		4	6.7361e−053	4.0001		5	9.5844e−154	4.0000
DF^1_{Sh}	0.15	50	1.1355e−010	n.c.	0.5	–	–	n.c.
DF^1_0		50	1.4909e−008	n.c.		–	–	n.c.
DF^1_1		50	8.6736e−009	n.c.		–	–	n.c.

Table 6 Convergence to the root $\alpha = 0$ of multiplicity 6 of equation f_6^6

Method	x_0	n	$\|x_n - x_{n-1}\|$	*ACOC*	x_0	n	$\|x_n - x_{n-1}\|$	*ACOC*
MR_{Sh}	−1.5	4	2.5849e−095	4.0000	1	4	9.8471e−100	4.0000
MR_0		4	1.5916e−095	4.0000		4	6.7101e−100	4.0000
MR_1		4	1.6571e−095	4.0000		4	6.9269e−100	4.0000
DF^2_{Sh}	−1.5	4	2.1691e−063	4.0001	1	4	6.2776e−095	4.0000
DF^2_0		4	1.9775e−063	4.0001		4	6.5722e−095	4.0000
DF^2_1		4	1.9928e−063	4.0001		4	6.5483e−095	4.0000
DF^1_{Sh}	−1.5	5	4.4796e−083	4.0000	1	5	3.3154e−175	4.0000
DF^1_0		5	3.8242e−083	4.0000		5	3.1921e−175	4.0000
DF^1_1		5	3.8745e−083	4.0000		5	3.2023e−175	4.0000

methods need slightly more iterations to converge, but then, they reach the predicted multiplicity.

5 Conclusions

We have presented a family of iterative methods for solving nonlinear equations with multiple roots and compared it with an existing method. The resulting methods are optimal because they reach fourth order of convergence and only use three function evaluations per step. Adding one functional evaluation, we obtain a family of derivative free iterative methods. The selected methods of the new family compare well with the fourth order method chosen as reference as we have seen in the numerical examples.

References

1. H.T. Kung, J.F. Traub, Optimal order of one-point and multi-point iteration. J. Assoc. Comput. Mach. **21**, 643–651 (1974)

2. W. Bi, H. Ren, Q. Wu, Three-step iterative methods with eighth-order convergence for solving nonlinear equations. J. Comput. Appl. Math. **255**, 105–112 (2009)
3. W. Bi, Q. Wu, H. Ren, A new family of eighth-order iterative methods for solving nonlinear equations. Appl. Math. Comput. **214**, 236–245 (2009)
4. A. Cordero, J.L. Hueso, E. Martínez, J.R. Torregrosa, New modifications of Potra-Pták's method with optimal fourth and eighth order of convergence. J. Comput. Appl. Math. **234**, 2969–2976 (2010)
5. E. Schröder, Über unendlich viele Algorithmen zur Auflösung der Gleichungen. Math. Ann. **2**, 317–365 (1870)
6. C. Chun, B. Neta, A third-order modification of Newtons method for multiple roots. Appl. Math. Comput. **211**, 474–479 (2009)
7. Y.I. Kim, S.D. Lee, A third-order variant of NewtonSecant method finding a multiple zero. J. Chungcheong Math. Soc. **23**(4), 845–852 (2010)
8. B. Neta, Extension of Murakamis high-order nonlinear solver to multiple roots. Int. J. Comput. Math. **8**, 1023–1031 (2010)
9. H. Ren, Q. Wu, W. Bi, A class of two-step Steffensen type methods with fourth-order convergence. Appl. Math. Comput. **209**, 206–210 (2009)
10. Q. Zheng, J. Wang, P. Zhao, L. Zhang, A Steffensen-like method and its higher-order variants. Appl. Math. Comput. **214**, 10–16 (2009)
11. S. Amat, S. Busquier, On a Steffensen's type method and its behavior for semismooth equations. Appl. Math. Comput. **177**, 819–823 (2006)
12. X. Feng, Y. He, High order iterative methods without derivatives for solving nonlinear equations. Appl. Math. Comput. **186**, 1617–1623 (2007)
13. A. Cordero, J.R. Torregrosa, A class of Steffensen type methods with optimal order of convergence. Appl. Math. Comput. doi:10.1016/j.amc.2011.02.067
14. F. Marvasti, A. Jain, Zero crossings, bandwidth compression, and restoration of nonlinearly distorted band-limited signals. J. Opt. Soc. Am. **A 3**, 651–654 (1986)
15. S. Anza, C. Vicente, B. Gimeno, V.E. Boria, J. Armendáriz, Long-term multipactor discharge in multicarrier systems. Physics of Plasmas **14**(8), 082–112 (2007)
16. J.L. Hueso, E. Martínez, C. Teruel, New families of iterative methods with fourth and sixth order of convergence and their dynamics, in *Proceedings of the 13th International Conference on Computational and Mathematical Methods in Science and Engineering, CMMSE 2013*, 24–27 June 2013
17. A. Cordero, J.R. Torregrosa, Low-complexity root-finding iteration functions with no derivatives of any order of convergence. J. Comput. Appl. Math. doi:10.10016/j.cam.2014.01.024 (2014)
18. J.R. Sharma, R. Sharma, Modified Jarratt method for computing multiple roots. Appl. Math. Comput. **217**, 878–881 (2010)